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On the Domain of Convergence and Poles of Complex *J*-Fractions

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Consider the infinite *J*-fraction

$$\cfrac{a_0^2}{z-b_0-\cfrac{a_1^2}{z-b_1-\cfrac{a_2^2}{z-b_2-\ddots}}}$$

where $a_n \in \mathbb{C}\setminus\{0\}$, $b_n \in \mathbb{C}$. Under very general conditions on the coefficients $\{a_n\}$, $\{b_n\}$, we prove that this continued fraction converges to a meromorphic function in $\mathbb{C}\setminus\mathbb{R}$. Such conditions hold, in particular, if $\lim_n \mathfrak{I}(a_n)=\lim_n \mathfrak{I}(b_n)=0$ and $\sum_{n\geqslant 0}(1/|a_n|)=\infty$ (or $\sum_{n\geqslant 0}(|b_n|/|a_n a_{n+1}|)=\infty$). The poles are located in the point spectrum of the associated tridiagonal infinite matrix and their order determined in terms of the asymptotic behavior of the zeros of the denominators of the corresponding partial fractions. © 1998 Academic Press

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1. INTRODUCTION

1. One of T. Stieltjes' ([11], see also [1]) most celebrated results establishes that the continued J-fraction

$$\cfrac{a_0^2}{z-b_0-\cfrac{a_1^2}{z-b_1-\cfrac{a_2^2}{z-b_2-\ddots}}} \tag{1}$$

where $a_n \in \mathbb{R}\backslash\{0\}$ and $b_n \in \mathbb{R}$, converges uniformly on each compact subset contained in $\{\Im(z) \neq 0\}$ to a holomorphic function if the determinate case holds (for the definition see [13, p. 99], and point **3** in Section 2 below). In this case, this is equivalent to saying that the associated infinite Jacobi matrix

$$G = \begin{pmatrix} b_0 & a_1 & 0 & \cdots \\ a_1 & b_1 & a_2 & \cdots \\ 0 & a_2 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2}$$

defines a unique selfadjoint operator, or that the corresponding moment problem has a unique solution.

The concept of determinacy or indeterminacy applies as well to J-fractions (1) such that $a_n \in \mathbb{C}\backslash\{0\}$ and $b_n \in \mathbb{C}$. Our object is to see what extension there is to Stieltjes' Theorem in this more general situation. The spectral properties of the operator defined by G play a central role in the description of the analytic properties of (1). This operator, which we properly define in Section 2, will also be denoted by G. As usual, $\sigma(G)$ (respectively $\sigma_p(G)$ and $\sigma_{ess}(G)$) denotes the spectrum (respectively, the point and essential spectrum) of G.

We shall prove that if the determinate case holds and

$$\lim_n \Im(a_n) = \lim_n \Im(b_n) = 0, \tag{3}$$

then $\sigma_p(G)$ consists of at most a denumerable set of isolated points in $\mathbb{C}\backslash\mathbb{R}$.

Moreover, we shall prove the following: *Assume that* (3) *takes place and that the determinate case holds. Then, the continued fraction* (1) *converges uniformly on each compact subset of* $\mathbb{C}\backslash(\mathbb{R} \cup \sigma_p(G))$ *to a holomorphic function with poles at each point of* $\sigma_p(G)\backslash\mathbb{R}$.

As will be shown, each pole of (1) "attracts" a specific number of zeros of the denominators of the corresponding partial fractions equal to the order of the pole. The rest of the zeros accumulate on $\mathbb{R}$ (for details see Theorem 2 below).

When the coefficients of G are real and the determinate case takes place then $\sigma(G) \subset \mathbb{R}$ (G is selfadjoint). Thus $\sigma_p(G) \setminus \mathbb{R} = \varnothing$ and we obtain Stieltjes' Theorem.

The proof of Theorem 2, which is carried out in Section 4, makes use of the asymptotic behavior of the zeros of polynomials satisfying three-term recurrence relations with complex coefficients. This question, which is interesting in itself, is studied in Section 3. We prove (see Theorem 1): *Assume that* (3) *takes place. Then, the zeros of the polynomials generated by the three-term recurrence relation*

$$\begin{aligned} &P_{n+1}(z) = (z - b_n)\, P_n(z) - a_n^2 P_{n-1}(z), \qquad n \geqslant 0, \\ &P_{-1}(z) = 0, \qquad P_0(z) = 1, \end{aligned} \tag{4}$$

accumulate exclusively on $\mathbb{R} \cup \sigma_p(G)$.

In Section 2, we introduce the notation and prove some auxiliary results.

2. AUXILIARY RESULTS

1. It is a well known fact (see [13, p. 197]) that there is a one to one correspondence between J-fractions (1) with $a_n \in \mathbb{C} \setminus \{0\}$ and $b_n \in \mathbb{C}$, and formal power series expansions at infinity

$$f(z) = \sum_{n \geqslant 0} \frac{c_n}{z^{n+1}} \tag{5}$$

such that

$$h_n = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & & \vdots \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix} \neq 0, \qquad n \geqslant 0. \tag{6}$$

In the sequel, f represents both the J-fraction and the associated formal power series.

The denominator P_n of the nth partial fraction

$$f_n(z) = \cfrac{a_0^2}{z - b_0 - \cfrac{a_1^2}{z - b_1 - \ddots - \cfrac{a_{n-1}^2}{z - b_{n-1}}}}$$

satisfies the recurrence relation (4) with the indicated initial conditions. The numerator Q_n satisfies the same recurrence relation but wit initial conditions $Q_0(z)=0$, $Q_1(z)=a_0^2$. These polynomials are also characterized by:

(i) $\deg P_n \leqslant n$, $\deg Q_n \leqslant n-1$, $P_n \not\equiv 0$,

(ii) $(P_n f - Q_n)(z) = A_n/z^{n+1} + \cdots$.

For the coefficients A_n, we have the formulas

$$A_n = a_0^2 a_1^2 \cdots a_n^2 = \frac{h_n}{h_{n-1}}, \qquad n \geqslant 0, h_{-1} = 1. \tag{7}$$

The sequence of rational functions $\{f_n\}$ is also called the main diagonal of the table of Padé approximants associated with f. Thus, the results we are about to establish may be stated as well for formal power series of type (5) for which (6) takes place. The polynomials P_n and Q_n are connected by the relations

$$(Q_{n+1} P_n - P_{n+1} Q_n)(z) = A_n. \tag{8}$$

These formulas, and others we shall use in the course of the paper, may be found, for example, in Chapters I, IV, V, and XX of [13]. Since they are well known (and easy to verify), we will use them without further reference.

2. Along with the sequences $\{P_n\}$ and $\{Q_n\}$, we make use of the so-called associated polynomials of type k. They are given by the recurrence relations

$$\begin{aligned} &P_{n+1}^{(k)}(z) = (z - b_{n+k})\, P_n^{(k)}(z) - a_{n+k}^2 P_{n-1}^{(k)}(z), \qquad n \geqslant 0, \\ &P_{-1}^{(k)}(z) = 0, \qquad P_0^{(k)}(z) = 1. \end{aligned} \tag{9}$$

This is the sequence of monic polynomials associated with the infinite tridiagonal matrix

$$G^{(k)} = \begin{pmatrix} b_k & a_{k+1} & 0 & \cdots \\ a_{k+1} & b_{k+1} & a_{k+2} & \cdots \\ 0 & a_{k+2} & b_{k+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to check that $P_n = P_n^{(0)}$ and $Q_n = a_0^2 P_{n-1}^{(1)}$. Thus

$$f_n = a_0^2 \frac{P_{n-1}^{(1)}}{P_n^{(0)}}. \tag{10}$$

Since a_0^2 is a common factor for f and all f_n there is no loss of generality if we assume in the sequel that $a_0 = 1$.

We also make use of the normalized associated polynomials. They are given by the formula

$$p_n^{(k)}(z) = \frac{P_n^{(k)}(z)}{a_k \cdots a_{k+n}}, \qquad n \geqslant 0, \tag{11}$$

and (9) can be rewritten in the following way

$$\begin{gathered} a_{n+k+1} p_{n+1}^{(k)}(z) = (z - b_{n+k})\, p_n^{(k)}(z) - a_{n+k} p_{n-1}^{(k)}(z), \qquad n \geqslant 1, \\ p_0^{(k)}(z) = \frac{1}{a_k}, \qquad p_1^{(k)}(z) = \frac{z - b_k}{a_k a_{k+1}}. \end{gathered} \tag{12}$$

This normalization slightly differs from the one adopted in [2, 3] but it allows us to give several formulas a closer form. Analogously,

$$q_n(z) = \frac{Q_n(z)}{a_0 \cdots a_n}, \qquad n \geqslant 0. \tag{13}$$

In general, $q_n(z) = a_0 p_{n-1}^{(1)}$, $n \geqslant 1$, but for $a_0 = 1$ they are equal. From (7), (8), and (13), we have

$$(q_{n+1} p_n - p_{n+1} q_n)(z) = \frac{1}{a_{n+1}}, \qquad n \geqslant 0. \tag{14}$$

3. As pointed out in the Introduction a basic concept in the convergence theory of continued fractions is that of determination.

Definition 1. The determinate case or the indeterminate case is said to hold for the continued fraction (1) (or for G) when at least one of the following two series diverges or both these series converge, respectively,

$$\sum_{k \geqslant 0} |p_k(z)|^2, \qquad \sum_{k \geqslant 0} |q_k(z)|^2,$$

at a given (fixed) point z of the complex plane.

A remarkable fact is the Theorem of Invariability (see Theorem 22.1 in [13]) which states that if at a given point both these series converge then at all points of the complex plane both converge. This theorem is what makes Definition 1 consistent.

Since determination plays a central role in this paper, we wish to underline the following sufficient condition proposed as Exercise 5.1 in [13].

LEMMA 1. *A sufficient condition for the determinate case to hold is that there exist $z \in \mathbb{C}$ and $m \in \mathbb{Z}_+$ such that*

$$\sum_{n \geqslant 0} |p_m^{(n+1)}(z)| = \infty. \tag{15}$$

In particular, this is true if either

$$\sum_{n \geqslant 0} \frac{1}{|a_{n+1}|} = \infty \qquad or \qquad \sum_{n \geqslant 0} \frac{|b_{n+1}|}{|a_{n+1} a_{n+2}|} = \infty. \tag{16}$$

Proof. The basic relation is the following generalization of (14)

$$(q_{n+m+1} p_n - p_{n+m+1} q_n)(z) = p_m^{(n+1)}(z). \tag{17}$$

The proof of (17) may be found in [12] (it is easy to verify by induction on m, notice that for $m = 0$ it reduces to (14)). The rest is a simple use of (17) and the Cauchy–Schwarz inequality. Relation (16) readily follows because

$$p_0^{(n+1)}(z) = \frac{1}{a_{n+1}} \qquad \text{and} \qquad p_1^{(n+1)}(z) = \frac{z - b_{n+1}}{a_{n+1} a_{n+2}}.$$

Thus we conclude the proof. ∎

4. Before proceeding let us take a closer look at (2). The infinite matrix G defines, by the usual operation on a vector, an operator on the linear subspace D_0 of l^2 formed by all vectors which have only a finite number of components different from zero. Along with D_0, we consider the linear subspace of l^2 given by $D_G = \{\bar{x} \in l^2 \colon G\bar{x} \in l^2\}$. D_G is called the maximal domain of G. A real tridiagonal symmetric matrix H defines an operator with a unique selfadjoint extension if it is a symmetric operator on D_H. For short, we say that H is selfadjoint. For real Jacobi matrices, selfadjointness and determination are equivalent.

When we refer to the spectrum of G (or part of it), we are considering G with domain D_G. In particular, $z \in \sigma_p(G)$ if and only if there exists $\bar{x} \in l^2$ such that $G\bar{x} = z\bar{x}$. From the three-term recurrence relation, it is easy to verify that

$$z \in \sigma_p(G) \Leftrightarrow \sum_{n \geqslant 0} |p_n(z)|^2 < +\infty. \tag{18}$$

LEMMA 2. *Let $k \in \mathbb{N}$ be given. The following statements are equivalent*:

(i) $G^{(k)}$ *is determinate.*

(ii) $\sigma_p(G^{(k)}) \cap \sigma_p(G^{(k+1)}) \neq \mathbb{C}$.

(iii) $\sigma_p(G^{(k)}) \cap \sigma_p(G^{(k+1)}) = \varnothing$.

(iv) $G^{(j)}$ *is determinate for all* $j \in \mathbb{N}$.

(v) $\bigcup_{j \geqslant 0} (\sigma_p(G^{(j)}) \cap \sigma_p(G^{(j+1)})) = \varnothing$.

(vi) $\bigcap_{j \geqslant 0} \sigma_p(G^{(j)}) \neq \mathbb{C}$.

Proof. The equivalence of (i) and (ii) is an immediate consequence of the definition of determination and (18) (as applied to $G^{(k)}$). In turn, that (ii) implies (iii) follows from the Theorem of Invariability, while the opposite implication is trivial.

In order to prove that (iv) follows from (i), we make use of the following well known relation (see, e.g., [2] or [12]):

$$a_j p_n^{(j)}(z) = (z - b_j)\, p_{n-1}^{(j+1)}(z) - a_{j+1} p_{n-2}^{(j+2)}(z). \tag{19}$$

Assume the contrary; that is, for some $j \in \mathbb{N}$, let $G^{(j)}$ be indeterminate. Then, there exists $z \in \mathbb{C}$ such that

$$\sum_{n \geqslant 0} |p_n^{(j)}(z)|^2 < +\infty, \qquad \sum_{n \geqslant 0} |p_n^{(j+1)}(z)|^2 < +\infty. \tag{20}$$

Therefore, using (19) and (20) consecutively (for larger and smaller indexes j), we obtain that for all $j \in \mathbb{N}$ the corresponding series converge. In particular, this would occur for k and $k+1$ and $G^{(k)}$ would be indeterminate. Obviously, (iv) implies (i).

Now, (iv) and (v) are equivalent because (i) is equivalent to (iii). Obviously, (ii) implies (vi). On the other hand, (vi) implies that for some $j \in \mathbb{N}$ $\sigma_p(G^{(j)}) \cap \sigma_p(G^{(j+1)}) \neq \mathbb{C}$ and using the equivalence of (i) with (ii) and (iv), we obtain that $G^{(k)}$ is determinate. ∎

Let us study other properties of the operator G.

LEMMA 3. *Consider the following decomposition of G*

$$G = H + C,$$

where H is a real tridiagonal symmetric matrix (*not necessarily defining a symmetric operator in D_H*). *Then*:

(i) *G admits such a decomposition, with C a bounded operator on* l^2, *if and only if*

$$\sup_{n \geqslant 1}\{|\mathfrak{I}(a_n)|, |\mathfrak{I}(b_n)|\} < +\infty.$$

(ii) *If C is bounded, then* $D_G = D_H$.

(iii) *If H is determinate, C is bounded, and* $z \in \sigma_p(G)$, *then*

$$d(z, \sigma(H)) \leqslant \|C\|,$$

where $d(z, \sigma(H))$ *denotes the distance between z and* $\sigma(H)$, *and* $\|C\|$ *the norm of the operator C acting on* l^2. *In particular, G is determinate.*

(iv) *If H is determinate and C is bounded, then for any other decomposition* $G = H_1 + C_1$, *where* H_1 *is a real tridiagonal symmetric matrix and* C_1 *is bounded,* H_1 *must be determinate.*

(v) *G admits such a decomposition, with C a compact operator on* l^2, *if and only if* (3) *takes place.*

(vi) *If H is determinate and C is compact, then* $\sigma_{ess}(G) = \sigma_{ess}(H)$ *and* $\sigma_p(G) \backslash \sigma_{ess}(H)$ *consists of at most a denumerable set of isolated points in* $\mathbb{C} \backslash \sigma_{ess}(H)$.

Proof. It is well known that a tridiagonal matrix defines a bounded operator on l^2 if and only if all its entries are uniformly bounded, and a compact one if and only if the diagonal entries tend to zero. Since $C = G - H$, where G and H are tridiagonal, then C is tridiagonal. On the other hand, H is real; therefore, the imaginary parts of the entries in C equal the imaginary parts of the entries in G. Hence, C bounded implies that the imaginary parts of the entries of G must be uniformly bounded, and C compact implies (3). The reciprocal statements in (i) and (v) are easy to deduce choosing H conveniently. It is even possible to construct H so that all its entries in the lower and upper diagonals are different from zero (as in G).

The assertion in (ii) is obvious because $C\bar{x} \in l^2$ for all $\bar{x} \in l^2$. In order to prove (iv), notice that $H_1 = H + C - C_1$. Therefore, $D_G = D_H = D_{H_1}$. Since H is a real, determinate J-matrix, the operator it defines is symmetric on $D_H = D_{H_1}$. On the other hand, $C - C_1$ is real and bounded; therefore, it defines a symmetric operator on $l^2 \supset D_{H_1}$. Therefore, H_1 defines a real symmetric operator on its maximal domain D_{H_1} and thus H_1 is determinate. The statements in (vi) follow from Weyl's Theorem on compact perturbation of selfadjoint operators (see [6]).

In order to prove (iii), take $z \in \sigma_p(G)$. If $z \in \sigma(H)$ the inequality is trivial. Thus, assume that $z \notin \sigma(H)$. Hence, $(H - zI)^{-1}$ is a bounded operator on

l^2 and $\|(H-zI)^{-1}\| = 1/d(z, \sigma(H))$. Take an eigenvector $\bar{x}$ of norm one corresponding to the eigenvalue z. We have

$$0 = (G - zI)\,\bar{x} = (H - zI)\,\bar{x} + C\bar{x},$$

or what is the same,

$$\bar{x} = -(H - zI)^{-1}\, C\bar{x}.$$

Taking norms on either side, we obtain

$$1 \leqslant \|(H - zI)^{-1}\|\ \|C\| = \frac{\|C\|}{d(z, \sigma(H))},$$

which is equivalent to the inequality we needed to prove. The final assertion in (iii) is immediate because $\sigma(H)$ is contained in the real line; therefore, (ii) in Lemma 2 takes place for $k = 0$. ∎

It would be nice to have a reciprocal of (iii). Some sufficient conditions are easy to prove.

LEMMA 4. *Let $G = H + C$, where H is a real symmetric tridiagonal matrix, C is bounded, and G is determinate. Then H is determinate if any one of the following conditions take place*:

(i) *Those in* (16).

(ii) $\sup_{n \geqslant 1} |b_n| < +\infty$.

(iii) $a_n \in \mathbb{R} \backslash \{0\}$.

Proof. Without loss of generality, we can assume that $\Re(a_n) \geqslant 0$, because in the construction of the monic polynomials P_n and Q_n the coefficients a_n appear to the square in the three-term recurrence relation. From (iv) of Lemma 3, it is sufficient to show that the lemma is true for a convenient decomposition of G. We do this taking H_1 with entries $\beta_n = \Re(b_n)$ on the main diagonal and $\alpha_n = |a_n|$ on the upper and lower diagonals. From the assumptions (see (i) in Lemma 3), we know that the imaginary parts of a_n and b_n are uniformly bounded. It is easy to deduce that the entries of $C_1 = G - H_1$ are uniformly bounded.

Assume that (i) takes place. If

$$\sum_{n \geqslant 0} \frac{1}{|a_{n+1}|} = \infty,$$

then H_1 is determinate. In particular, this settles the problem if there is a subsequence of $\{a_n\}$ which is bounded. Therefore, we may restrict our attention to the case when $\lim_n |a_n| = \infty$, $\sum_{n\geqslant 0} 1/|a_{n+1}| < \infty$, and

$$\sum_{n\geqslant 0} \frac{|b_n|}{|a_n a_{n+1}|} = \infty.$$

From these conditions it immediately follows that

$$\sum{}' \frac{|b_n|}{|a_n a_{n+1}|} = \infty,$$

where $\sum'$ stands for the sum taken over those indexes n such that $|\Re(b_n)| \geqslant |\Im(b_n)|$. In fact, the contrary would imply that

$$\sum_{n\geqslant 0} \frac{|b_n|}{|a_n a_{n+1}|} \leqslant 2c_1 \sum_{n\geqslant 0} \frac{1}{|a_{n+1}|} + \sum{}' \frac{|b_n|}{|a_n a_{n+1}|} < \infty,$$

where $c_1 = \sup_n |\Im(b_n)|/\inf_n |a_n|$. Therefore,

$$\infty = \sum{}' \frac{|b_n|}{|a_n a_{n+1}|} \leqslant 2 \sum_{n\geqslant 0} \frac{|\Re b_n|}{|a_n a_{n+1}|},$$

which implies that H_1 is determinate.

Conditions (ii) and (iii) imply that H_1 is determinate by use of the Theorem of Invariability (see [13, p. 96]) after carrying out equivalence transformations of the continued fraction in order to substitute a_n^2 by $|a_n^2|$. ∎

Remark 1. It may be proved that condition (15) on G is sufficient to guarantee that H is determinate.

5. Another important concept in the theory of continued fractions is that of positive definiteness. For the proper definition, we refer the reader to [13, p. 67]. We state the following equivalent form (see Corollary 16.2 in [13]).

DEFINITION 2. The J-fraction (1) is positive definite if, and only if,

(a) $\beta_n = \Im(b_n) \leqslant 0$, $n \in \mathbb{N}$.

(b) there exist numbers $g_0, g_1, \ldots$, such that $0 \leqslant g_{n-1} \leqslant 1$ and

$$\alpha_n^2 \leqslant \beta_n \beta_{n+1} (1 - g_{n-1})\, g_n, \qquad n \in \mathbb{N},$$

where $\alpha_n = \Im a_n$.

The following lemma is a consequence of Theorem 25.4 in [13].

LEMMA 5. *Let $d>0$. Assume that G is determinate and*

$$\sup_{n\geqslant 0}\{|\mathfrak{I}(b_n)|, 2\,|\mathfrak{I}(a_{n+1})|\}\leqslant d.$$

Then,

$$\lim_n f_n = f, \qquad \{|\mathfrak{I}(z)|>2d\}, \tag{21}$$

uniformly on each compact subset of the indicated region.

Proof. First, let us concentrate on the set $\{\mathfrak{I}(z)>2d\}$. In place of (1), consider the J-fraction

$$\cfrac{a_0^2}{z'-(b_0-2d)-\cfrac{a_1^2}{z'-(b_1-2d)-\cfrac{a_2^2}{z'-(b_2-2d)-\ddots}}} \tag{22}$$

where $z'=z-2d$.

With respect to z', (22) is positive definite. In fact,

$$\mathfrak{I}(b_n-2d)\leqslant -d<0, \qquad n\in\mathbb{N},$$

and taking $g_n=\frac{1}{2}$, we have

$$(\mathfrak{I}(a_n))^2\leqslant\frac{d^2}{4}\leqslant\mathfrak{I}(b_n-2d)\,\mathfrak{I}(b_{n+1}-2d)(1-g_{n-1})\,g_n, \qquad n\in\mathbb{N}.$$

On the other hand, by the Theorem of Invariability, the J-fraction (22) is determinate.

Therefore, Theorem 25.4 in [13] asserts that (22) converges uniformly on each compact subset of $\{\mathfrak{I}(z')>0\}$ to a holomorphic function. That is to say that our initial J-fraction (1) satisfies (21) in $\{\mathfrak{I}(z)>2d\}$.

For the lower part, notice that the continued fraction

$$\cfrac{1}{z-\bar{b}_0-\cfrac{\bar{a}_1^2}{z-\bar{b}_1-\ddots-\cfrac{\bar{a}_n^2}{z-\bar{b}_n}-\ddots}}$$

also satisfies the conditions of the theorem regarding the sequences of its coefficients $\{\bar{b}_n\}$ and $\{\bar{a}_n\}$. It is easy to see that the corresponding partial fractions f_n^* are linked with those of f_n by the formula

$$f_n^*(\bar{z}) = \overline{f_n(z)}, \qquad n \geqslant 0,$$

and we only have to use the result we have just proved for the upper half to the sequence $\{f_n^*\}$. ∎

3. LOCATION OF ZEROS

1. Let

$$\mathscr{P}(G) = \{z\colon \exists \Lambda \subset \mathbb{N},\ P_n(z_n) = 0,\ \forall n \in \Lambda,\ z = \lim_{n \in \Lambda} z_n\}.$$

In the sequel, H always represents a real, tridiagonal, symmetric matrix. By Favard's Theorem, the polynomials generated by the three-term recurrence relation associated with H are orthogonal with respect to some measure μ_H, whose support, $\operatorname{supp} \mu_H$, is contained in the real line. This measure is not uniquely determined, unless H is determinate. By μ_H, we denote any particular solution. We have:

THEOREM 1. *Assume that G admits a decomposition $G = H + C$, where C is bounded. Denote*

$$A = \bigcap_{H = G - C,\ C \text{ bounded}} \{z\colon d(z, \operatorname{conv}(\operatorname{supp} \mu_H)) \leqslant \inf_k \|C^{(k)}\|\},$$

where $\operatorname{conv}(\operatorname{supp} \mu_H)$ *denotes the convex hull of the indicated set. Then*

$$\mathscr{P}(G) \subset \sigma_p(G) \cup A.$$

Proof. If G is indeterminate, then $\sigma_p(G) = \mathbb{C}$ (see Lemma 2) and the statement is trivial. Therefore, the main interest is when G is determinate.

Let $z \in \mathscr{P}(G)$ and take $\Lambda \subset \mathbb{N}$ such that

$$z = \lim_{n \in \Lambda} z_n,$$

where $P_n(z_n) = 0$ for all $n \in \Lambda$; define

$$w^n = (p_0(z_n), p_1(z_n), \ldots, p_{n-1}(z_n), 0, 0, \ldots) \in l^2.$$

For

$$L = \overline{\lim_{n \in \Lambda}} \|w^n\|, \tag{23}$$

we can consider two cases:

(i) $L < \infty$. Then the sequence $\{w^n\}_{n \in \Lambda}$ is uniformly bounded and there exists a weakly convergent subsequence $\{w^n\}_{n \in \Lambda'}$, $\Lambda' \subset \Lambda$ (see, e.g., [5]). Denote by $w \in l^2$ its weak limit; since each component $p_i(z_n)$ of w^n, $n \in \Lambda'$, converges to $p_i(z)$, it follows that $w = (p_0(z), p_1(z), ...) \in l^2$. Then (12) (for $k = 0$) gives us $(G - zI)\, w = 0$, $w \neq 0$, which implies that $z \in \sigma_p(G)$.

(ii) $L = \infty$. Then, there exists $\Lambda' \subset \Lambda$ such that

$$\lim_{n \in \Lambda'} \|w^n\| = \infty. \tag{24}$$

Given $n \in \Lambda$ and $k \in \mathbb{N}$, $1 \leqslant k \leqslant n-1$, define

$$w^{k,n} = (p^{(k+1)}_{n-k-1}(z_n), p^{(k+2)}_{n-k-2}(z_n), ..., p^{(n)}_0(z_n)) \in \mathbb{C}^{n-k}.$$

Making use of the relations

$$a_k p^{(k)}_n(z) = (z - b_k)\, p^{(k+1)}_{n-1}(z) - a_{k+1} p^{(k+2)}_{n-2}(z) \tag{25}$$

(see [12]), it is easy to verify that

$$(G^{(k)}_{n-k} - z_n I_{n-k})\, w^{k,n} = -a_k p^{(k)}_{n-k}(z_n)\, e^0_{n-k},$$

where $G^{(k)}_{n-k}$ is the principal section of order $n-k$ of $G^{(k)}$, I_{n-k} is the identity matrix of order $n-k$, and e^0_{n-k} is the $n-k$ dimensional vector with 1 on its first component and the rest equal to zero. Taking

$$u^{k,n} = \frac{w^{k,n}}{\|w^{k,n}\|},$$

where

$$\|w^{k,n}\| = \left(\sum_{j=1}^{n-k} |p^{(n-j+1)}_{j-1}(z_n)|^2 \right)^{1/2},$$

we arrive at

$$(G^{(k)}_{n-k} - z_n I_{n-k})\, u^{k,n} = \frac{-a_k p^{(k)}_{n-k}(z_n)}{(\sum_{j=1}^{n-k} |p^{(n-j+1)}_{j-1}(z_n)|^2)^{1/2}}\, e^0_{n-k}. \tag{26}$$

In [2, Lemma 3], the following equation was established

$$p_{n-j}(z_n) = a_n p_{j-1}^{(n-j+1)}(z_n)\, p_{n-1}(z_n), \qquad j = 1, ..., n \tag{27}$$

(recall that here the associated polynomials are normalized differently than in [2]). In particular, for $j = n-k+1$, we obtain

$$p_{n-k}^{(k)}(z_n) = \frac{p_{k-1}(z_n)}{a_n p_{n-1}(z_n)}. \tag{28}$$

Using (27) and (28) in (26), we have

$$\|(G_{n-k}^{(k)} - z_n I)\, u^{k,n}\|^2 = \frac{|a_k p_{k-1}(z_n)|^2}{\sum_{j=k}^{n-1} |p_j(z_n)|^2}.$$

Taking the limit $n \to \infty$, $n \in \Lambda'$, and considering (24), we conclude that

$$\lim_{n \in \Lambda'} \|(G_{n-k}^{(k)} - z_n I)\, u^{k,n}\| = 0.$$

Moreover, for each $n \in \Lambda'$, we have $\|u^{k,n}\| = 1$; therefore,

$$\|(G_{n-k}^{(k)} - zI)\, u^{k,n}\| \leqslant \|(G_{n-k}^{(k)} - z_n I)\, u^{k,n}\| + |z - z_n|,$$

and

$$\lim_{n \in \Lambda'} \|(G_{n-k}^{(k)} - zI)\, u^{k,n}\| = 0. \tag{29}$$

Consider any decomposition of G of the form $G = H + C$, where C is bounded. Take $G_{n-k}^{(k)} = H_{n-k}^{(k)} + C_{n-k}^{(k)}$ (where $H_{n-k}^{(k)}$, $C_{n-k}^{(k)}$ are the principal sections of order $n-k$ of $H^{(k)}$ and $C^{(k)}$, respectively). Then

$$(H_{n-k}^{(k)} - zI)\, u^{k,n} = (G_{n-k}^{(k)} - zI)\, u^{k,n} - C_{n-k}^{(k)} u^{k,n}.$$

In case that $z \notin \sigma(H_{n-k}^{(k)})$ $(= \sigma_p(H_{n-k}^{(k)}))$, we may apply $(H_{n-k}^{(k)} - zI)^{-1}$ to both sides of the previous equation and obtain

$$u^{k,n} = (H_{n-k}^{(k)} - zI)^{-1} [(G_{n-k}^{(k)} - zI) - C_{n-k}^{(k)}]\, u^{k,n}.$$

Thus,

$$1 \leqslant \|(H_{n-k}^{(k)} - zI)^{-1}\| \, [\, \|(G_{n-k}^{(k)} - zI)\, u^{k,n}\| + \|C_{n-k}^{(k)}\| \,]. \tag{30}$$

Since $H_{n-k}^{(k)}$ is a finite real symmetric matrix, we have that

$$d(z, \sigma(H_{n-k}^{(k)})) = \frac{1}{\|(H_{n-k}^{(k)} - zI)^{-1}\|}.$$

Therefore, from (30) we obtain

$$d(z, \sigma(H_{n-k}^{(k)})) \leqslant \|(G_{n-k}^{(k)} - zI)\, u^{k,n}\| + \|C_{n-k}^{(k)}\|. \tag{31}$$

If $z \in \sigma(H_{n-k}^{(k)})$, then (31) evidently holds true.

Using (29), the well-known fact that

$$\lim_{n\to\infty} \|C_{n-k}^{(k)}\| = \|C^{(k)}\|,$$

and taking limits in (31), we have

$$\overline{\lim_{n\in\Lambda'}}\, d(z, \sigma(H_{n-k}^{(k)})) \leqslant \|C^{(k)}\|, \qquad \forall k.$$

From this inequality it is easy to conclude the proof. In fact, $\sigma(H_{n-k}^{(k)})$ coincides with the set of zeros of the associated polynomial of type k and degree $n-k$ relative to H. It is well known that these zeros all lie in $\operatorname{conv}(\operatorname{supp}\mu_H)$. Therefore,

$$d(z, \operatorname{conv}(\operatorname{supp}\mu_H)) \leqslant d(z, \sigma(H_{n-k}^{(k)})),$$

and from the limit above, we obtain

$$d(z, \operatorname{conv}(\operatorname{supp}\mu_H)) \leqslant \|C^{(k)}\|, \qquad \forall k.$$

Now, it is sufficient to take inf on k in order to conclude the proof. ∎

Of special interest is the following

COROLLARY 1. *Let $G = H + C$, where C is compact. Then*

$$\mathscr{P}(G) \subseteq \sigma_p(G) \cup \operatorname{conv}(\operatorname{supp}\mu_H).$$

Proof. Since C is compact, then

$$\lim_{n\to\infty} \|C^{(k)}\| = 0,$$

and the result is immediate from Theorem 1. ∎

4. CONVERGENCE OF CONTINUED FRACTIONS

1. Now, we can prove the extension of Stieltjes' Theorem. Before stating the result, we will introduce some more notation.

Along with the continued fraction (1), which we denoted by $f(z)$, we will consider the continued fractions

$$f^{(k)}(z)=\cfrac{1}{z-b_k-\cfrac{a_{k+1}^2}{z-b_{k+1}-\ddots-\cfrac{a_{k+n}^2}{z-b_{k+n}}-\ddots}}$$

The corresponding nth partial fraction is

$$f_n^{(k)}(z)=\cfrac{1}{z-b_k-\cfrac{a_{k+1}^2}{z-b_{k+1}-\ddots-\cfrac{a_{k+n-1}^2}{z-b_{k+n-1}}}}$$

Notice that

$$f_n^{(k)}(z)=\frac{P_{n-1}^{(k+1)}(z)}{P_n^{(k)}(z)}. \tag{32}$$

2. Fix $\zeta\in\mathbb{C}$ and let $\{z_{n-i}\}$, $i=1, ..., n$, be the set of zeros of P_n. Assume that they are indexed so that

$$|z_{n,1}-\zeta|\leqslant|z_{n,2}-\zeta|\leqslant\cdots\leqslant|z_{n,n}-\zeta|.$$

Definition 3. Let $\zeta\in\mathbb{C}$. We denote by $\underline{\lambda}(\zeta)$ the number $j\in\mathbb{N}$ such that

$$\varliminf_n |z_{n,j}-\zeta|=0 \qquad \text{and} \qquad \varliminf_n |z_{n,j+1}-\zeta|>0.$$

If $\varliminf_n |z_{n,1}-\zeta|>0$, we take $\underline{\lambda}(\zeta)=0$. On the other hand, if $\varliminf_n |z_{n,j}-\zeta|=0$ for all $j\in\mathbb{N}$, then $\underline{\lambda}(\zeta)=+\infty$.

For $\zeta=\infty$ (here, $|z_{n,n}|\leqslant|z_{n,n+1}|\leqslant\cdots\leqslant|z_{n,1}|$), $\underline{\lambda}(\zeta)$ equals $j\in\mathbb{N}$ if

$$\varlimsup_n |z_{n,j}|=+\infty \qquad \text{and} \qquad \varlimsup_n |z_{n,j+1}|<+\infty.$$

When $\varlimsup_n |z_{n,j}|=+\infty$ for all $j\in\mathbb{N}$ then $\underline{\lambda}(\infty)=+\infty$ and $\underline{\lambda}(\infty)=0$ if $\varlimsup_n |z_{n,1}|<+\infty$.

A similar characteristic was introduced by A. A. Gonchar in [9].

DEFINITION 4. Given $\zeta \in \mathbb{C}$, we denote $\lambda(\zeta)$ the number $j \in \mathbb{N}$ such that

$$\lim_n |z_{n,j} - \zeta| = 0 \qquad \text{and} \qquad \overline{\lim_n} |z_{n,j+1} - \zeta| > 0.$$

If $\overline{\lim}_n |z_{n,1} - \zeta| > 0$ then $\lambda(\zeta) = 0$. In case that $\lim_n |z_{n,j} - \zeta| = 0$ for all $j \in \mathbb{N}$, then $\lambda(\zeta) = +\infty$.

For $\zeta = \infty$, $\lambda(\zeta)$ equals $j \in \mathbb{N}$ if

$$\lim_n |z_{n,j}| = +\infty \qquad \text{and} \qquad \underline{\lim_n} |z_{n,j+1}| < +\infty.$$

When $\lim_n |z_{n,j}| = +\infty$ for all $j \in \mathbb{N}$, then $\lambda(\infty) = +\infty$. If $\underline{\lim}_n |z_{n,1}| < +\infty$, then $\lambda(\infty) = 0$.

Obviously, $\lambda(\zeta) \leqslant \underline{\lambda}(\zeta)$ for each fixed $\zeta \in \overline{\mathbb{C}}$. If $\{z_{n,i}\}$, $i = 1, ..., n$, denotes the set of zeros of $P_n^{(1)}$, we define similarly $\underline{\lambda}^{(1)}(\zeta)$ and $\lambda^{(1)}(\zeta)$, respectively.

Another important index is $\kappa(P_n, U)$ (see [7, 8]).

DEFINITION 5. Let U be an open set of the extended complex plane. By $\kappa(P_n, U)$, we denote the number of zeros of P_n contained in U.

These indices play an important role in the study of the so-called inverse problems in the theory of Padé approximants. For a review on the subject, see [10] and also the references above. Here, they are used to describe the connection between the asymptotic distribution of the zeros of $\{P_n\}$, $n \in \mathbb{Z}_+$, and the poles of $f(z)$.

We also define $\kappa(\zeta)$ as follows:

DEFINITION 6. Assume that f is meromorphic on a certain region D, and $\zeta \in D$. Then $\kappa(\zeta) = m$ if ζ is a zero of multiplicity m. If ζ is a pole of order p, then $\kappa(\zeta) = -p$. Otherwise, $\kappa(\zeta) = 0$.

3. We are ready for the proof of

THEOREM 2. *Assume that G is determinate, $G = H + C$, and C is compact. Then, $\sigma_p(G) \backslash \mathbb{R}$ consists of at most a denumerable set of isolated points in $\{|\mathfrak{I}(z)| \neq 0\}$ and*

$$\sigma_p(G) \backslash \mathbb{R} = \mathscr{P}(G) \backslash \mathbb{R}.$$

The continued fraction (1) *converges uniformly on each compact subset of $\{|\mathfrak{I}(z)| \neq 0\} \backslash \sigma_p(G)$ to a holomorphic function which is meromorphic in $\{|\mathfrak{I}(z)| \neq 0\}$. Moreover, we have*:

(i) *For* $\zeta \in \sigma_p(G^{(1)})\backslash\mathbb{R}$, *there exist* $\rho > 0$ *and* $n_1(\rho)$ *such that*

$$\kappa(Q_n, D_\rho(\zeta)) = \underline{\lambda}^{(1)}(\zeta) = \lambda^{(1)}(\zeta) = \kappa(\zeta) \geqslant 1, \qquad n \geqslant n_1(\rho), \tag{33}$$

where $D_\rho(\zeta) = \{z: |z-\zeta| < \rho\}$.

(ii) *For* $\zeta \in \sigma_p(G)\backslash\mathbb{R}$, *there exist* $\rho > 0$ *and* $n_2(\rho)$ *such that*

$$-\kappa(P_n, D_\rho(\zeta)) = -\underline{\lambda}(\zeta) = -\lambda(\zeta) = \kappa(\zeta) \leqslant -1, \qquad n \geqslant n_2(\rho). \tag{34}$$

Proof. Let $d > 0$. Choose and fix an integer $k \geqslant 0$ such that

$$\sup_{n \geqslant 0}\{|\Im(b_{k+n})|, 2\,|\Im(a_{k+n+1})|\} \leqslant d.$$

This is possible because C is a compact operator (see (v) in Lemma 3).

By use of Lemma 5, we obtain that

$$f_n^{(k)}(z) \underset{n}{\rightrightarrows} f^{(k)}(z), \qquad \{|\Im(z)| > 2d\}, \tag{35}$$

where $f^{(k)}$ is holomorphic in $\{|\Im(z)| > 2d\}$. Notice that (see [13, p. 15])

$$f_{k+n}(z) = \cfrac{a_0^2}{z - b_0 - \cfrac{a_1^2}{z - b_1 - \ddots - \cfrac{a_{k-1}^2}{z - b_{k-1} - a_k^2 f_n^{(k)}(z)}}}$$

$$= \frac{Q_k(z) - a_k^2 Q_{k-1}(z)\, f_n^{(k)}(z)}{P_k(z) - a_k^2 P_{k-1}(z)\, f_n^{(k)}(z)} = \frac{Q_{k+n}(z)}{P_{k+n}(z)}. \tag{36}$$

From (35), we obtain that

$$P_k(z) - a_k^2 P_{k-1}(z)\, f_n^{(k)}(z) \underset{n}{\rightrightarrows} P_k(z) - a_k^2 P_{k-1}(z)\, f^{(k)}(z) \tag{37}$$

and

$$Q_k(z) - a_k^2 Q_{k-1}(z)\, f_n^{(k)}(z) \underset{n}{\rightrightarrows} Q_k(z) - a_k^2 Q_{k-1}(z)\, f^{(k)}(z), \tag{38}$$

uniformly on each compact subset of $\{|\Im(z)| > 2d\}$. The functions

$$P_k(z) - a_k^2 P_{k-1}(z)\, f^{(k)}(z) \qquad \text{and} \qquad Q_k(z) - a_k^2 Q_{k-1}(z)\, f^{(k)}(z)$$

are holomorphic in $\{|\Im(z)| > 2d\}$.

Using (36)–(38), we have that $\{f_n\}$, $n \geqslant 0$, converges uniformly on each compact subset of $\{|\Im(z)| > 2d\} \cap \{P_k(z) - a_k^2 P_{k-1}(z)\, f^{(k)}(z) \neq 0\}$ to the holomorphic function

$$f(z) = \frac{Q_k(z) - a_k^2 Q_{k-1}(z)\, f^{(k)}(z)}{P_k(z) - a_k^2 P_{k-1}(z)\, f^{(k)}(z)}. \tag{39}$$

Let $\Omega(d)$ be the largest open set contained in $\{|\Im(z)| > 2d\}$ where $\{f_n\}$, $n \geqslant 0$, converges uniformly on each compact subset.

We have that

$$\Omega(d) \supset \{|\Im(z)| > 2d\} \cap \{P_k(z) - a_k^2 P_{k-1}(z)\, f^{(k)}(z) \neq 0\}.$$

Let us show that $\Omega(d)$ equals $\{|\Im(z)| > 2d\}$ except for a denumerable set of isolated points. Obviously, it is sufficient to prove that $\{P_k(z) - a_k^2 P_{k-1}(z)\, f^{(k)}(z) = 0\}$ consists of isolated points in $\mathbb{C}$.

Assume that for some $z_0 \in \mathbb{C}$

$$P_k(z_0) - a_k^2 P_{k-1}(z_0)\, f^{(k)}(z_0) = 0.$$

We have that $P_{k-1} \neq 0$, because P_k and P_{k-1} may not have common zeros (see (7) and (8)). Then

$$f^{(k)}(z_0) = \frac{P_k(z_0)}{a_k^2 P_{k-1}(z_0)}.$$

Therefore, if z_0 is not an isolated zero, we have that

$$f^{(k)}(z) \equiv P_k(z)/a_k^2 P_{k-1}(z)$$

is a rational function. This is not possible because by Kronecker's Theorem (for example, see Theorem 51.2 in [13]) that would imply that for all sufficiently large n the Hankel determinants $h_n^{(k)}$ corresponding to $f^{(k)}$ would have to be zero, which is not the case because $a_j \neq 0$, for all $j \in \mathbb{N}$ (see Subsection 2.1, in particular (7), as regards to the continued fraction $f^{(k)}$).

Let us prove that

$$\Omega(d) \subset \{|\Im(z)| > 2d\} \setminus \sigma_p(G) \subset \{|\Im(z)| > 2d\} \setminus \mathscr{P}(G).$$

The second set is contained in the third because of Corollary 1. Let z_0 belong to the first set. Then,

$$\lim_n \frac{q_n(z_0)}{p_n(z_0)} = f(z_0) \in \mathbb{C}.$$

If $z_0 \in \sigma_p(G)$, from this limit, one immediately obtains that $z_0 \in \sigma_p(G^{(1)})$ which is absurd because G is determinate (see Lemma 2). Therefore, the first set is contained in the second. In particular, we have that the sets $\sigma_p(G)\backslash\{|\Im(z)|>2d\}$ and $\mathscr{P}(G)\backslash\{|\Im(z)|>2d\}$ consist of isolated points.

Now, let us prove that

$$\{|\Im(z)|>2d\}\backslash\mathscr{P}(G)\subset\Omega(d).$$

Let $z_0 \in \{|\Im(z)|>2d\}\backslash\mathscr{P}(G)$. Since $\mathscr{P}(G)\backslash\{|\Im(z)|>2d\}$ and the subset of $\{|\Im(z)|>2d\}$ where the continued fraction diverges only have isolated points, we can find $\rho>0$ sufficiently small such that for all sufficiently large n, the disk $\overline{D_\rho(z_0)}$ contains no zeros of p_n and the sequence $\{f_n\}$, $n\geqslant 0$ converges uniformly on the boundary of this disk. But then, by the maximum principle of analytic functions, we have uniform convergence in all $\overline{D_\rho(z_0)}$. Thus, $z_0 \in \Omega(d)$ as we claimed.

We conclude that

$$\Omega(d)=\{|\Im(z)|>2d\}\backslash\sigma_p(G)=\{|\Im(z)|>2d\}\backslash\mathscr{P}(G).$$

Since $d>0$ may be taken arbitrarily small, it follows that

$$\sigma_p(G)\backslash\mathbb{R}=\mathscr{P}(G)\backslash\mathbb{R},$$

$\sigma_p(G)\backslash\mathbb{R}$ consists of isolated points in $\{|\Im(z)|\neq 0\}$, and

$$f_n(z)\underset{n}{\rightrightarrows} f(z), \qquad \{|\Im(z)|\neq 0\}\backslash\sigma_p(G),$$

where $f(z)$ is holomorphic in $\{|\Im(z)|\neq 0\}\backslash\sigma_p(G)$. In $\{|\Im(z)|\neq 0\}$, f is meromorphic because in a reduced neighborhood of each point in $\sigma_p(G)\backslash\mathbb{R}$ it is expressed as the quotient of two holomorphic functions (see (39) for k sufficiently large).

We have proved all the initial statements of the theorem. Before proceeding we wish to make two remarks. The first is that the zeros of f must be isolated points in $\{|\mathfrak{F}(z)|\neq 0\}$. The contrary would lead to a contradiction using again Kronecker's Theorem. The second is that, for all $j\in\mathbb{N}$, $\sigma_p(G^{(j)})\backslash\mathbb{R}$ consists solely of isolated points and

$$\sigma_p(G^{(j)})\backslash\mathbb{R}=\mathscr{P}(G^{(j)})\backslash\mathbb{R}.$$

This is a consequence of what we have just proved considering the continued fraction $f^{(j)}$ in place of f (see (iv) in Lemma 2). We are especially interested in the second remark for $j=1$.

Fix $\zeta\in\{|\Im(z)|\neq 0\}$. From what we have proved so far, we can find $\rho>0$ sufficiently small with the following properties: the set $\overline{D_\rho(\zeta)}\backslash\{\zeta\}$ contains no points belonging to $\sigma_p(G)\cup\sigma_p(G^{(1)})\cup\mathbb{R}$ and has no zero or pole of f.

If ρ satisfies the properties above, so does any ρ' such that $0<\rho'\leqslant\rho$. Fix ρ with the specified property. Now, we can find $n_0(\rho)$ such that for $n\geqslant n_0(\rho)$ the circumference $C_\rho(\zeta)=\{|z-\zeta|=\rho\}$ does not contain zeros of P_n or Q_n.

From the uniform convergence of $\{f_n\}$, $n\geqslant 0$, on $C_\rho(\zeta)$, we obtain

$$\lim_n \int_{C_\rho(\zeta)} \frac{f_n'(z)}{f_n(z)}\,dz=\int_{C_\rho(\zeta)} \frac{f'(z)}{f(z)}\,dz.$$

By the argument principle, and the fact that the integrals on which we are taking limit represent integers, we obtain that for all $n\geqslant n_0'(\rho)\geqslant n_0(\rho)$

$$\kappa(Q_n, D_\rho(\zeta))-\kappa(P_n, D_\rho(\zeta))=\kappa(\zeta). \tag{40}$$

From Lemma 2, we know that $\sigma_p(G^{(1)})\cap\sigma_p(G)=\varnothing$. Therefore, if $\zeta\in\sigma_p(G^{(1)})\backslash\mathbb{R}$, then for all $n\geqslant n_1(\rho)\geqslant n_0'(\rho)$, we have that $\kappa(P_n, D_\rho(\zeta))=0$. Therefore, (40) states that

$$\kappa(Q_n, D_\rho(\zeta))=\kappa(\zeta), \qquad n\geqslant n_1(\rho).$$

Since the same is true for all ρ' such that $0<\rho'<\rho$, we conclude that

$$\kappa(Q_n, D_\rho(\zeta))=\underline{\lambda}^{(1)}(\zeta)=\lambda^{(1)}(\zeta)=\kappa(\zeta), \qquad n\geqslant n_1(\rho).$$

For (33) it remains to prove that $\kappa(\zeta)\geqslant 1$. This immediately follows from the fact that

$$\sigma_p(G^{(1)})\backslash\mathbb{R}=\mathscr{P}(G^{(1)})\backslash\mathbb{R}.$$

By Lemma 2 again, if $\zeta\in\sigma_p(G)\backslash\mathbb{R}$, then for all $n\geqslant n_2(\rho)\geqslant n_0'(\rho)$, we obtain that $\kappa(Q_n, D_\rho(\zeta))=0$ and (40) reduces to

$$-\kappa(P_n, D_\rho(\zeta))=\kappa(\zeta), \qquad n\geqslant n_1(\rho).$$

Since the same is true for all ρ' such that $0<\rho'<\rho$, then we have that

$$-\kappa(P_n, D_\rho(\zeta))=-\underline{\lambda}(\zeta)=-\lambda(\zeta)=\kappa(\zeta), \qquad n\geqslant n_2(\rho).$$

In order to complete (34), we must prove that $\kappa(\zeta)\leqslant -1$ for $\zeta\in\sigma_p(G)\backslash\mathbb{R}$. This is a consequence of the equality

$$\sigma_p(G)\backslash\mathbb{R}=\mathscr{P}(G)\backslash\mathbb{R}.$$

With this we conclude the proof of Theorem 2. ∎

4. We wish to underline some consequences of Theorem 2.

Corollary 2. *Assume that* (3) *and* (15) *hold. Then the theses of Theorem* 2 *are valid.*

Proof. Using (v) from Lemma 3, we have that $G = H + C$, where H is a real tridiagonal symmetric matrix and C is compact. From Lemma 1, we know that G is determinate. Therefore, we have the conditions required in Theorem 2. ∎

Of special interest for its practical expression is the following particular case of Corollary 2.

Corollary 3. *Assume that* (3) *and* (16) *hold. Then the theses of Theorem* 2 *are valid.*

Proof. As noted in Lemma 1, (16) implies (15). Thus, the conditions of Corollary 2 are fulfilled. ∎

We wish to conclude this list of applications of Theorem 2 with the following extension of Markov's Theorem on the convergence of continued fractions (Padé approximants of Markov-type functions).

Corollary 4. *Assume that* $\sup_n |a_n| \leqslant M < +\infty$ *and* (3) *takes place. Then the theses of Theorem* 2 *hold.*

Proof. It is obvious that in this situation $\sum_n 1/|a_n| = \infty$ and we may use Corollary 3. ∎

Markov's Theorem refers to continued fractions corresponding to the Cauchy transform of a positive Borel measure supported on a finite interval of the real line. In this case, it is easy to prove that $0 < a_n \leqslant M$ and $-M \leqslant b_n \leqslant M$, where M is a positive constant independent of n.

By means of equivalence transformations, we can obtain an analogue of Theorem 2 for S-fractions. To see how this is done and the type of result which may be obtained we refer the reader to Corollaries 4 and 5 of [2].

Remark 2. We wish to point out that in (34) (respectively (33)) $\kappa(\zeta)$ equals minus the algebraic multiplicity of ζ as an eigenvalue of G (respectively the algebraic multiplicity of ζ as an eigenvalue of $G^{(1)}$). The proof of these statements is contained in a forthcoming paper (see [4]) where we study the analogue of the results above for general band matrices.

5. To conclude, we will state a generalization of Theorem 2.

THEOREM 3. *Let G be determinate. Suppose that there exist $b \in \mathbb{C}$ and $\theta \in [0, \infty)$ such that*

$$\lim_{n \to \infty} \Im([b_n - b]\, e^{i\theta}) = 0, \qquad \lim_{n \to \infty} \Im(a_n e^{i\theta}) = 0.$$

Denote by l the straight line of the complex plane defined by

$$l = \{z : (z - b)\, e^{i\theta} \in \mathbb{R}\}.$$

Then, $\sigma_p(G) \backslash l$ consists of at most a denumerable set of isolated points in $\mathbb{C} \backslash l$ and

$$\sigma_p(G) \backslash l = \mathscr{P}(G) \backslash l.$$

The continued fraction (1) *converges uniformly on each compact subset of $\mathbb{C} \backslash (\sigma_p(G) \cup l)$ to a holomorphic function which is meromorphic in $\mathbb{C} \backslash l$. Moreover, we have*

(i) *For $\zeta \in \sigma_p(G^{(1)}) \backslash l$, there exist $\rho > 0$ and $n_1(\rho)$ such that*

$$\kappa(Q_n, D_\rho(\zeta)) = \underline{\lambda}^{(1)}(\zeta) = \lambda^{(1)}(\zeta) = \kappa(\zeta) \geqslant 1, \qquad n \geqslant n_1(\rho), \tag{41}$$

where $D_\rho(\zeta) = \{z : |z - \zeta| < \rho\}$.

(ii) *For $\zeta \in \sigma_p(G) \backslash l$, there exist $\rho > 0$ and $n_2(\rho)$ such that*

$$-\kappa(P_n, D_\rho(\zeta)) = -\underline{\lambda}(\zeta) = -\lambda(\zeta) = \kappa(\zeta) \leqslant -1, \qquad n \geqslant n_2(\rho). \tag{42}$$

Proof. We will not go into detail in the proof because though the statement is more general than that of Theorem 2 in fact these two results are equivalent. This is due to the fact that J-fractions (diagonal Padé approximants), the point spectrum of a Jacobi matrix (as well as the other components of its spectrum), and determinacy are invariant under an affine transformation. For more details on how to carry on, we refer the reader to Subsection 3.2 of [2] and Subsection 2.1 of [3]. ∎

6. We express our gratitude to a referee who pointed out that Lemma 5 could be deduced from Theorem 25.4 in [13] as we have done in fact. That, and calling our attention indirectly to the notion of determinacy of a continued fraction, has resulted in a considerable improvement of the paper. Initially, we had only considered compact perturbations of selfadjoint operators. In this regard, we proved (see Lemma 4 and

Remark 1) that for a wide class of Jacobi matrices, if G is determinate then so is H. We think that this is always the case whenever $G = H + C$ with C bounded.

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